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Symbolic calculus on the nilpotent orbits of $SO_0(1,2)$

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Abstract

The coadjoint conical orbits in $so(1,2)^* \simeq su(1,1)^*$ are the phase spaces of the zero mass particles on the two-dimensional (anti-)de Sitter space–time. It contains also, as an open dense subset, the phase space for massless particles on one-dimensional Minkowski space–time when one identifies the Poincaré group to a subgroup of the conformal group $SO_0(2,2) \simeq SO_0(1,2) \times SO_0(1,2)$. On the other hand, the quantum representation associated to these systems is an indecomposable extension of the first term of the discrete series of representations of $SO_0(1,2) \simeq SU(1,1)/\mathbb{Z}_2$. We present in this paper a symbol map linking this representation and this orbit. This calculus is invariant and behaves correctly in the classical limit. As a result we have obtained a conformally invariant symbolic calculus for massless particles on (anti-)de Sitter or Minkowski space–time in 1+1 dimension.

Keywords: Coadjoint conical orbits; Zero mass particles; Symbolic calculus; Poincaré group; (Anti-)de Sitter space–time; Minkowski space–time

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1. Introduction

This paper presents an invariant quantization of the nilpotent orbits (which are conical) of $SO_0(1,2) \simeq SU(1,1)/\mathbb{Z}_2$, by means of a symbol map on the first term of the discrete series of representations of $SO_0(1,2)$, U^1 and on its one-dimensional extension V^+. This symbol map is defined on the representations of the universal enveloping algebra of $so(1,2)$ generated by U^1 and V^+. In the following, we shall identify $SO_0(1,2)$ with $SU(1,1)/\mathbb{Z}_2$. The representations of $SU(1,1)$ involved in this paper are also representations of $SO_0(1,2)$.

Since the works of Weyl and Wigner, the problem of quantization, that is to say the correspondence between classical (symplectic manifold) and quantum (operator algebra)

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description has been treated by many authors in different ways, for instance: Berezin quantization using coherent states [Be], Stratonovitch correspondence [FGV,GMNO] and Poisson algebra deformations [BF^{2}LS]. This correspondence, called symbolic calculus, is realized by means of a symbol map from a class of operators on the quantum Hilbert space to a class of functions on the symplectic manifold. This symbol map has to verify some nice properties like group invariance and classical limit property that we shall detail later. In this framework, we consider two problems which are already open.

- The conical coadjoint orbits in so$(1,2)^*$ are phase space for massless particles on (anti-)de Sitter space–time in two dimension. In [DBR1], we construct the quantum representation by means of a geometric construction: orbit method modulo dilations. By this construction, we obtained the correct quantization, namely, the unique one-dimensional extension V^+ of the first term U^1 of the discrete series representation of $\mathrm{SU}(1,1)/\mathbb{Z}_2 \simeq \mathrm{SO}_0(1,2)$. It is then natural to look for a calculus, with symbols living on the cone, on the representation of the universal enveloping algebra generated by U^1 and V^+. On one hand, there exists a calculus with symbols living on the cone [U], but this calculus is related to the metaplectic representation of the universal covering of $\mathrm{SO}_0(1,2)$ and not to U^1, moreover the method of deformations failed in quantizing the cone [Fr]. On the other hand, there exist also symbolic calculi on U^1 [Be,ACG,UU], but there, the symbols live on the elliptic orbits which are associated to massive particles.
- The finite-dimensional conformal group associated to the two-dimensional Minkowski space–time is the group $\mathrm{SO}_0(2,2) \simeq \mathrm{SO}_0(1,2)\times\mathrm{SO}_0(1,2)$. One can verify that only one term of this product acts nontrivially on right (or left) moving particles. Moreover, the corresponding coadjoint orbit is also the cone (in fact a dense open subset of the cone if we consider only the Poincaré group) [R] and the quantum representation is the same as above [DBR2]. Symbolic calculi have been constructed for the massless representation of the Poincaré group in one [R] and three [ACM] dimension but they are not conformally (i.e. $\mathrm{SO}_0(2,2)$) invariant and hence not satisfactory.

In this paper, we solve these two problems by constructing an invariant symbol map for both the representations V^+ and U^1 of the universal enveloping algebra of SU(1, 1) *with symbols living in the conical orbit.* This furnishes an invariant symbolic calculus for (anti-)de Sitter massless particles, which is automatically conformally invariant (see [DBR1]), and a conformally invariant symbolic calculus for the massless representation of the Poincaré group (see [DBR2]).

The dilations of su$(1,1)^*$ play a crucial role in the construction of V^+ [DBR1]. On one hand, the quotient (diffeomorphic to S^1) of each cone by dilations is isomorphic (as homogeneous space) to the set of right (or left) moving lightlike geodesics on (anti-)de Sitter space–time. On the other hand, modulo dilations, the conical orbits are the boundary of the elliptic orbits. The representation V^+ is realized on functions on S^1 which are boundary values of holomorphic functions on the elliptic orbits, which can be identified as homogeneous space to the unit disk with the homographic action of SU(1,1). The fact that the representation V^+ can be viewed in some sense as the boundary of the representation U^1 associated to an elliptic orbit will play a crucial role in the construction of our calculus, *our symbols will be defined essentially as boundary values of Berezin symbols on the disk.*

It is somehow surprising that the operators algebra which we obtained is not a deformation of the full Poisson algebra but of a subalgebra. Actually, this is not *very* surprising so far as the coadjoint orbit is not a phase space because there is no classical description of massless particles. The physically relevant classical objects are space–time geodesics which are related, by the so-called "schema de Souriau", not to the orbit but to the orbit modulo dilations [R], hence it is not surprising that classical observables must verify some homogeneity properties.

The rest of this paper is organized as follows. In Section 2, we recall some useful facts about the construction of the representation V^+ and prove that eventually we can forget V^+ and consider only U^1. In Section 3, we define the symbol of an operator and prove its properties in Section 4.

2. Quantization of a SU(1, 1) nilpotent orbit

We consider the Lie group $\mathrm{SU}(1,1)/\{\pm \mathrm{Id}\}$, the following matrices form a basis for its complexified Lie algebra su(1, 1):

$$N_o = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad N_+ = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ \mathrm{i} & 0 \end{pmatrix}, \qquad N_- = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Among the orbits of $\mathrm{SU}(1,1)/\{\pm \mathrm{Id}\}$ in its coadjoint action, we consider the nilpotent orbit C defined by $N_o^2 = N_+N_-$, $N_o > 0$ (in this equation the elements of su(1, 1) are identified with their natural corresponding ones in the bidual su(1, 1)**). We choose polar coordinates on the cone C, $(\lambda, t) \in \mathbb{R}^{+*} \times \mathbb{R}/2\pi\mathbb{Z}$ which are canonical, the 2-form reads $\omega = \mathrm{d}\lambda \wedge \mathrm{d}t$. In these coordinates the action of the group reads:

$$g \cdot (\lambda, t) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot (\lambda, t) = \left(\lambda|\alpha + \beta \mathrm{e}^{\mathrm{i}t}|^2,\ -\arg\left(\frac{\alpha \mathrm{e}^{-\mathrm{i}t} + \beta}{\bar{\beta}\mathrm{e}^{-\mathrm{i}t} + \bar{\alpha}} \right) \right) \tag{2}$$

for g in $\mathrm{SU}(1,1)/\{\pm \mathrm{Id}\}$. The hamiltonian generators corresponding to the above basis (1) of su(1, 1) are:

$$k_o(\lambda, t) = \lambda, \qquad k_+(\lambda, t) = \lambda \mathrm{e}^{-\mathrm{i}t}, \qquad k_-(\lambda, t) = \lambda \mathrm{e}^{\mathrm{i}t}.$$

This orbit is nilpotent. For this reason, there are some difficulties in applying orbit method to it. It has been shown in [DBR1] that the orbit method modulo dilations allows a quantization of this orbit, and this quantization is compatible with the physical interpretations described in [DBR1,DBR2]. By this way, we have obtained an indecomposable representation V^+ of $\mathrm{SU}(1,1)/\{\pm \mathrm{Id}\}$ on a space $\mathcal{H}^+$. This representation is an extension of the first term of the discrete series of representations by the trivial representation on $\mathbb{C}$. That is to say, we have the following exact sequence of su(1,1)-modules:

$$0 \to (\mathbb{C}, U^0) \xrightarrow{i} (\mathcal{H}^+, V^+) \xrightarrow{D} (\mathcal{B}^1, U^1) \to 0, \tag{3}$$

in which $(\mathcal{B}^1, U^1)$ is the first term of the discrete series, to be defined more precisely in the next section. The space $\mathcal{H}^+$ can be realized as a space of functions on C/dilations $\sim S^1$, and it is connected with $\mathcal{B}^1$ by a boundary limit process. Actually, any element $\psi(t)$ in $\mathcal{H}^+$ is the boundary value of a (unique) holomorphic function in the unit disk, denoted by $\psi(z)$ as well, and the operator D in (3) is the operator $\mathrm{d}/\mathrm{d}z$. This construction is realized in a geometric way described in detail in [DBR1].

Moreover, the representations U^1 and V^+ induce representations (on Gårding subspaces) of the enveloping algebra $\mathcal{U} = \mathcal{U}(\mathrm{su}(1,1))$ and for our purpose, that is constructing a symbolic calculus, we can forget V^+ and consider only U^1. Indeed, for any operator A in $V^+(\mathcal{U})$, we construct the operator $\tilde{A} = DAD^{-1}$ in $U^1(\mathcal{U}) =: \mathfrak{A}$ which is well defined because $A\,(i(\mathbb{C})) \subset i(\mathbb{C})$. One can verify easily, using (3), that this correspondence is a bijection commuting with addition, multiplication, adjointness and the action of $\mathrm{SU}(1,1)/\{\pm\,\mathrm{Id}\}$ (adjoint representation).

Hence, we have only to construct a symbol map

$$\begin{aligned} \mathfrak{A} &\longrightarrow C^\infty(C) \\ A &\longmapsto A(\lambda, t). \end{aligned}$$

However, the construction of V^+ and its link with U^1 explain the crucial role of the boundary limit in the construction of the symbol.

Note that the orbit defined by $N_o^2 = N_+N_-$, $N_o < 0$, is associated to the first term of the antiholomorphic discrete series which is (anti)isomorphic to U^1. Hence, this case can be treated in a very similar way.

3. Definition of the symbol

Let us recall that the first Bargmann space is:

$$\mathcal{B}^1 = \{f : \mathcal{D} \to \mathbb{C};\ f \text{ holomorphic and } f \in L^2(\mathcal{D}, (1/\pi)\mathrm{d}z\,\mathrm{d}\bar{z})\},$$

where $\mathrm{d}z\,\mathrm{d}\bar{z}$ is the Lebesgue measure on the disk $\mathcal{D} = \{z \in \mathbb{C};\ |z| < 1\}$. The representation, which, from now on, we shall denote by U instead of U^1, is defined by

$$\left(U_g f\right)(z) = (-\bar{\beta}z + \alpha)^{-2} f\left(\frac{\bar{\alpha}z - \beta}{-\bar{\beta}z + \alpha}\right) \quad \text{for } g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

The infinitesimal generators corresponding to the basis (1) read:

$$K_o = \hbar z\frac{\mathrm{d}}{\mathrm{d}z} + \hbar, \qquad K_+ = \hbar z^2\frac{\mathrm{d}}{\mathrm{d}z} + 2\hbar z, \qquad K_- = \hbar\frac{\mathrm{d}}{\mathrm{d}z}.$$

Remark on the appearance of $\hbar$: In the case of a nilpotent orbit, there is only one character from which we induce the representation in the orbit method, the trivial one. Hence, we obtain only one representation V^+, and hence U, which does *not* depend on $\hbar$. This factor appears only by the way of the usual formula

$$K_j = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\theta}U_{e^{\theta N_j}}, \quad j \in \{o, +, -\}.$$

This explains the above dependence in $\hbar$ of the generators, which is in contrast with the case of the discrete series of representations of $SO_0(1,2)$ interpreted as a relativity group for massive particles on curved space–time [R,GR].

For this representation, there exist symbolic calculi with symbols living on the disk which is an orbit for $SU(1,1)/\{\pm \mathrm{Id}\}$, for instance the Berezin calculus defined by using covariant coherent states:

$$e_{z_o}(z) = (1-\bar{z}_o z)^{-2} \in \mathcal{B}^1.$$

We then define the symbol by [Be]:

$$A(z,\bar{z}) = \frac{(e_{z_o}, Ae_{z_o})}{(e_{z_o}, e_{z_o})}.$$

These symbols are covariant in the sense that

$$U_g^{-1}AU_g(z,\bar{z}) = A(g\cdot z, \overline{g\cdot z}),$$

where $g \cdot z$ is the natural action of $SU(1,1)/\{\pm \mathrm{Id}\}$ on $\mathcal{D}$. Moreover, we have

$$A^*(z,\bar{z}) = \overline{A(z,\bar{z})},$$

and

$$\mathrm{Id}(z,\bar{z}) = 1, \qquad K_o(z,\bar{z}) = \hbar\frac{1+z\bar{z}}{1-z\bar{z}},$$

$$K_+(z,\bar{z}) = \hbar\frac{z}{1-z\bar{z}}, \qquad K_-(z,\bar{z}) = \hbar\frac{\bar{z}}{1-z\bar{z}}.$$

Now we are looking for a symbolic calculus with symbols living on the cone and not on the disk. For this purpose, we begin with decomposing the adjoint action, denoted by "ad", of $su(1,1)$ on $\mathfrak{A}$. We call $\mathfrak{A}_n$ the vectorial subspace of $\mathfrak{A}$ generated by the $K_o^\alpha K_+^\beta K_-^\gamma$ with $\alpha+\beta+\gamma \leq n$. It is easy to see that $\mathfrak{A}_n$ is a $su(1,1)$-submodule of $\mathfrak{A}$. Moreover, we have

$$\mathfrak{A}_n = \mathfrak{A}_{n-1} + \mathrm{span}\langle K_+^p K_o^q, K_o^n, K_o^q K_-^p;\ p+q=n, 0<p\leq n\rangle.$$

This is a consequence of the following relations obtained by means of both the relations of commutation and the fact that the Casimir operator vanishes in this representation:

$$\begin{aligned} &[K_-, K_o] = \hbar K_-, \qquad [K_o, K_+] = \hbar K_+, \\ &K_-K_+ = K_o^2 + \hbar K_o, \qquad K_+K_- = K_o^2 - \hbar K_o. \end{aligned} \tag{4}$$

Note that the two last relations and their consequences $K_+K_- = K_o^2 + O(\hbar)$, etc. are closely related to our particular case. In order to have the same notations as [Bo], let

$$H = \frac{2}{\hbar}K_o, \qquad X_+ = \frac{1}{\hbar}K_+, \qquad X_- = \frac{1}{\hbar}K_-.$$

The element K_+^n is primitive for the $su(1,1)$-module $\mathfrak{A}_n$ since we have

$$[X_+, K_+^n] = 0, \qquad [H, K_+^n] = 2nK_+^n.$$

As a consequence, the submodule V_n generated by K_+^n is, as $\mathbb{C}$-vector space, of dimension $2n+1$, and the following family is a basis for V_n:

$$E_p^{(n)} := (\mathrm{ad}_{X_-})^p K_+^n, \quad p = 0, \ldots, 2n. \tag{5}$$

Moreover, the following relations hold:

$$\begin{aligned} &[H, E_p^{(n)}] = 2(n-p)E_p^{(n)}, \\ &[X_-, E_p^{(n)}] = E_{p+1}^{(n)}, \\ &[X_+, E_p^{(n)}] = -p(2n-p+1)E_{p-1}^{(n)}. \end{aligned} \tag{6}$$

Hence, for dimensional reasons we have

$$\mathfrak{A}_n = \mathfrak{A}_{n-1} \oplus V_n \quad \text{and} \quad \mathfrak{A} = \bigoplus_{n=0}^{\infty} V_n.$$

Now, we can define the symbol of one operator in the following manner: for $\mathfrak{A} \ni A = \sum_n A_n$, $A_n \in V_n$, we define

$$A(\lambda, t) = \sum_n \left(\frac{\lambda}{\hbar}\right)^n \frac{1}{(n+1)!} \lim_{\substack{z=re^{-it} \\ r\to 1}} (1 - z\bar{z})^n A_n(z, \bar{z}). \tag{7}$$

The above limit is well defined because one can verify (see Appendix A) that for $A \in \mathfrak{A}_n$ we have

$$A_n(z, \bar{z}) = \frac{P(z, \bar{z})}{(1 - z\bar{z})^n},$$

where P is some polynomial expression. We do insist on the crucial role of boundary limit in this construction. The surprising fact is that the covariance property is not lost in this construction, thanks to formula (2).

4. Properties of the calculus

Theorem 4.1. *The transformation*

$$\begin{aligned} \mathfrak{A} &\to C^\infty(C) \\ A &\mapsto A(\lambda, t) \end{aligned}$$

defined above (7) *has the following properties*:

(1) $\mathrm{Id}(\lambda, t) = 1$ *and* $K_j(\lambda, t) = k_j(\lambda, t)$ *for* $j = o, +, -$.

(2) *The calculus is invariant*:

$$\left(U_g^{-1} A U_g\right)(\lambda, t) = A(g \cdot (\lambda, t)) \quad \forall g \in \mathrm{SU}(1,1), \ \forall A \in \mathfrak{A}.$$

(3) *The symbol map* $A \mapsto A(\lambda, t)$ *is injective.*

(4) $A^*(\lambda, t) = \overline{A(\lambda, t)}$ $\forall A \in \mathfrak{A}$.

(5) *For all A, B in $\mathfrak{A}$ and (λ, t) in C, the classical limit has the right properties*

$$\lim_{\hbar\to 0}(AB)(\lambda,t) = \left(\lim_{\hbar\to 0} A(\lambda,t)\right)\left(\lim_{\hbar\to 0} B(\lambda,t)\right),$$

$$\lim_{\hbar\to 0}\frac{1}{\mathrm{i}\hbar}[A,B](\lambda,t) = \left\{\lim_{\hbar\to 0} A(\lambda,t),\ \lim_{\hbar\to 0} B(\lambda,t)\right\},$$

where $\{\ ,\ \}$ is the Poisson bracket on C.

Remark. We can construct a star-product indexed by $\hbar$ as the pullback on $C^\infty(C)$ of the operators product. The fifth property shows that this product is a deformation, in the sense of [BF^{2}LS], of the Poisson product on C restricted to some subspace of $C^\infty(C)$. The second property proves that this star-product is invariant. Note that the quantizable observables are those polynomials in $\lambda, \mathrm{e}^{\mathrm{i}t}$ for which the degree in λ is greater than or equal to the degree in $\mathrm{e}^{\mathrm{i}t}$ (see Section 1 for a brief discussion).

Proof of Theorem 4.1. The first property is clear. The subspaces V_n are su(1, 1)-submodules, hence it is enough to prove the second property on V_n. Let A be in V_n, then we have

$$A(z,\bar z) = \frac{P(z,\bar z)}{(1-z\bar z)^n},$$

where P is some polynomial expression. The Bargmann calculus is covariant and we obtain for g as in (2),

$$U_g^{-1}AU_g(z,\bar z) = \frac{P(g\cdot z, \overline{g\cdot z})}{(1-g\cdot z\overline{g\cdot z})^n} = |\alpha+\beta\bar z|^{2n}\frac{P(g\cdot z,\overline{g\cdot z})}{(1-z\bar z)^n}.$$

Hence, the second property follows immediately from (2). In order to verify injectivity, we infer from Appendix A that

$$K_+^n(\lambda,t) = \lambda^n\mathrm{e}^{-\mathrm{i}nt} = k_+^n(\lambda,t).$$

Moreover, $C^\infty(C)$ has also a su(1,1)-module structure for which k_+^n is a primitive element with the same weight than K_+^n. Hence, the two submodules generated by K_+^n and k_+^n arc isomorphic. Thanks to the coefficients λ^n in the definition, the global injectivity is clear as well. As a byproduct of that precedes, we see that

$$E_p^{(n)}(\lambda,t) = \frac{(2n)!}{(2n-p)!}\lambda^n\mathrm{e}^{-\mathrm{i}(n-p)t}. \tag{8}$$

For adjointness property, we begin with the following proposition.

Proposition 4.2. *The properties $A\in\mathfrak{A}_n$, $[X_-,A]=0$ and $[H,A]=-2nA$ characterize A up to a multiplicative constant.*

Proof. We begin with calculation in $\mathfrak{A}_n/\mathfrak{A}_{n-1}$. Let

$$A=\sum_{p=1}^{n}\alpha_p K_+^p K_o^{n-p}+\gamma K_o^n+\sum_{p=1}^{n}\beta_p K_o^{n-p}K_-^p \text{ [modulo } \mathfrak{A}_{n-1}].$$

Then using the relations listed in Appendix A, we obtain

$$\begin{aligned}[X_-,A]=&\sum_{p=2}^{n}(n+p)\alpha_p K_+^{p-1}K_o^{n-p+1}+(n+1)\alpha_1 K_o^n\\ &+n\gamma K_o^{n-1}K_-+\sum_{p=2}^{n}(n-p+1)\beta_{p-1}K_o^{n-p}K_-^p \text{ [modulo } \mathfrak{A}_{n-1}].\end{aligned}$$

Hence, $[X_-,A]=0 \Rightarrow A=\beta_n K_-^n$ [modulo $\mathfrak{A}_{n-1}$]. Using again the same argument on $A-\beta_n K_-^n$, etc., we obtain that $A=\sum_{p=0}^{n}\beta_p K_-^p$, moreover, by a direct calculation [Bo], we have $[H,K_-^p]=-2pK_-^p$, then $[H,A]=-2nA$ implies that $A=\beta_n K_-^n$. □

Corollary 4.3. $E_{2n}^{(n)}=(2n)!K_-^n$.

Proof. The relation $E_{2n}^{(n)}=\beta_n K_-^n$ is a consequence of both (6) and Proposition 4.2. Moreover, from (4) we deduce easily that $E_{2n}^{(n)}=(2n)!K_-^n$ [modulo $\mathfrak{A}_{n-1}$], the result follows. □

Corollary 4.4. *The space V_n is invariant under the operation of taking adjoints.*

Proof. From Corollary 4.3, we obtain $(E_0^{(n)})^*=(1/(2n)!)E_{2n}^{(n)}$ and by induction, using (5),

$$(E_p^{(n)})^*=\frac{p!}{(2n-p)!}E_{2n-p}^{(n)}.$$ □

Proof of Theorem 4.1 (continued). Now the fourth property of the theorem follows easily from the fact that the same property holds for the Berezin symbols.

Finally, we have to prove the classical limit properties. From (4) we obtain

$$E_p^{(n)}=\begin{cases}\dfrac{(2n)!}{(2n-p)!}K_+^{n-p}K_o^p+\mathrm{O}(\hbar) & \text{for } 0\le p\le n,\\[2ex] \dfrac{(2n)!}{(2n-p)!}K_o^{2n-p}K_-^{p-n}+\mathrm{O}(\hbar) & \text{for } n\le p\le 2n.\end{cases} \tag{9}$$

Let

$$F_p^{(n)}=\begin{cases}K_+^{n-p}K_o^p & \text{for } 0\le p\le n,\\ K_o^{2n-p}K_-^{p-n} & \text{for } n\le p\le 2n.\end{cases}$$

Then from (8) and (9) we obtain

$$F_p^{(n)}(\lambda,t)=\lambda^n \mathrm{e}^{-\mathrm{i}(n-p)t}+\mathrm{O}(\hbar), \tag{10}$$

moreover, the algebra $\mathfrak{A}$ is commutative modulo $\hbar$, hence

$$F_{p+q}^{(n+m)} = F_p^{(n)} F_q^{(m)} + \mathrm{O}(\hbar),$$

from which we deduce by linearity the first property of the classical limit. Using Appendix A, we obtain also

$$[F_p^{(n)}, F_q^{(m)}] = \hbar(pm - qn) F_{p+q-1}^{(n+m-1)} + \mathrm{o}(\hbar). \tag{11}$$

The second property is an easy consequence of (10) and (11). The theorem is proved. □

Appendix A: Some calculations on Bargmann space operators

From (4), we obtain by direct calculation the following results which have been used freely in the end of this paper:

$$\begin{aligned}
&[K_-^q, K_o^p] = qp\hbar K_o^{p-1} K_-^q + \mathrm{o}(\hbar) = qp\hbar K_-^q K_o^{p-1} + \mathrm{o}(\hbar),\\
&[K_o^p, K_+^q] = qp\hbar K_+^q K_o^{p-1} + \mathrm{o}(\hbar),\\
&K_-^n K_+^n = K_o^{2n} + \hbar n^2 K_o^{2n-1} + \mathrm{o}(\hbar),\\
&K_+^n K_-^n = K_o^{2n} - \hbar n^2 K_o^{2n-1} + \mathrm{o}(\hbar).
\end{aligned}$$

On the other hand, we have [GR]:

$$(A_1 A_2)(z, \bar{z}) = \sum_{p \geq 0} \frac{1}{p!(p+1)!} \frac{\partial^p}{\partial \bar{w}^p} A_1\left(z, \frac{\bar{z} - \bar{w}}{1 - z\bar{w}}\right) \frac{\partial^p}{\partial w^p} A_2\left(\frac{z - w}{1 - \bar{z}w}, \bar{z}\right)\Bigg|_{w=\bar{w}=0}.$$

Hence, we have:

$$\begin{aligned}
K_o A(z, \bar{z}) &= \hbar\left(\frac{1 + z\bar{z}}{1 - z\bar{z}} A(z, \bar{z}) + z \frac{\partial A}{\partial z}(z, \bar{z})\right),\\
K_+ A(z, \bar{z}) &= \hbar\left(\frac{2z}{1 - z\bar{z}} A(z, \bar{z}) + z^2 \frac{\partial A}{\partial z}(z, \bar{z})\right),\\
K_- A(z, \bar{z}) &= \hbar\left(\frac{2\bar{z}}{1 - z\bar{z}} A(z, \bar{z}) + \frac{\partial A}{\partial z}(z, \bar{z})\right).
\end{aligned}$$

As a first consequence, we have

$$K_+^n(z, \bar{z}) = \hbar^n (n+1)! \frac{z^n}{(1 - z\bar{z})^n},$$

as a second consequence, we can prove by induction that for any $A \in \mathfrak{A}_n$ we have

$$A(z, \bar{z}) = \frac{P(z, \bar{z})}{(1 - z\bar{z})^n},$$

where P is some polynomial expression.

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